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Geometric phases and coherent states

Timothy R. Field [a,*], Jeeva S. Anandan [b,c,†]

[a] QinetiQ, Malvern Technology Park, St. Andrews Road, Malvern WR14 3PS, UK
[b] Department of Physics and Astronomy, University of South Carolina, Columbia SC 29208, USA
[c] Clarendon Laboratory, University of Oxford, Oxford OX1 3PU, UK

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Abstract

A cyclic evolution of a pure quantum state is characterized by a closed curve γ in the projective Hilbert space $\mathbb{P}\mathcal{H}$, equipped with the Fubini-Study geometry. It is known that the geometric phase $\vartheta_{\mathcal{G}}$ for this evolution is given by the integral of the symplectic form of the Fubini-Study geometry over an arbitrary surface spanning γ. This result extends to an infinite-dimensional Hilbert space for a bosonic quantum field. We prove that $\vartheta_{\mathcal{G}}$ is bounded above by the infimum area over all surfaces spanning γ, and that the bound is attained if γ can be spanned by a holomorphic curve. Using an earlier result concerning the intrinsic Euclidean geometry of the coherent state submanifold $\mathcal{C}$, we derive an expression for the geometric phase for a cyclic evolution amongst coherent states. We indicate how the intensity of a classical configuration can be inferred from the winding number of the exponential geometric phase about the origin in the complex plane. In the case of photon states we present group theoretic and 2-component spinor representations of $\vartheta_{\mathcal{G}}$. We derive an expression for $\vartheta_{\mathcal{G}}$ in the case of a sequence of measurements such that the resulting states are coherent at each step, in terms of a sequence of projection operators. The situation in relation to some earlier experiments of Pancharatnam and Tomita–Chiao is explained.
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1. Introduction

Geometric phases in physics have been of theoretical and experimental interest since Berry introduced the concept of a gauge invariant phase acquired by a quantum system

* Corresponding author.
E-mail address: trfield@signal.qinetiq.com (T.R. Field).
† Deceased.

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undergoing cyclic evolution, in the adiabatic approximation [12]. A precursor of Berry's work was due to [40] who introduced a geometric phase for classical electromagnetic fields, which correspond to coherent states of the quantized electromagnetic field. Berry's phase factor was interpreted geometrically as the holonomy of a connection on the parameter space of the Hamiltonian generating the cyclic evolution [43]. Aharonov and Anandan [1] and Anandan and Aharonov [8], extended this phase to non-adiabatic cyclic evolutions. The idea was extended further within the framework of *geometric quantum mechanics* (see [9,7,21–23,39]), in terms of a connection on a fiber bundle over the projective Hilbert space and a corresponding group theoretic treatment [3].

The subject of 'geometric quantum mechanics' [14], introduced in [17,18] and developed in a physical context by Kibble [30,31], describes the operations of the quantum theory in a geometrical language, in such a way that all the genuine degrees of freedom of the quantum system are manifest. This development should be viewed as distinct from geometric quantization which, in contrast, addresses the question of how to quantize classical systems by carrying over the symplectic structure present in their Hamiltonian description into the quantum domain [46]. In the latter process certain ambiguities arise, in the introduction of the metrical and associated complex structures that characterize a quantum system, but nevertheless certain natural quantization procedures have been found for specific kinds of systems.

Geometric quantum mechanics, on the other hand, may be viewed as the inverse construction. The approach taken here is to begin with a quantum mechanical system, equipped with dynamical (i.e. symplectic) and metrical structures with respect to some Kähler manifold $\mathcal{K}$, and attempt to formulate the operations of quantum theory in geometrical language. One then obtains a certain 'classical limit' by analyzing the space of coherent states, which are the preferred class of states from which an ancestral classical configuration can be derived in the form of the solution to some classical field equation, e.g. Maxwell's equations.

The state space that we shall consider throughout is complex projective space $\mathbb{CP}^n$, arising from the space of rays in the Hilbert space $\mathcal{H}$ of complex dimension $n+1$. In this way the geometrical description passes from the usual Hilbert space description to one in terms of the projective Hilbert space $\mathbb{P}\mathcal{H} = \mathbb{CP}^n$, with respect to which the dynamical and probabilistic aspects of the standard quantum theory can be represented. This subject has been developed by a number of authors in a variety of contexts. In particular the geometric formulation for a finite-dimensional quantum system was generalized by Field [22], Field and Hughston [23] to the field theoretic context, where the relevant Hilbert space is *Fock space* $\mathcal{F}$ (see e.g. [24,45,46]). In this situation the coherent states have a special role and their geometry has been studied in detail [6,23]. The more general state spaces $\mathcal{K}$ have been studied in the literature (e.g. [19,25,31]), and recently in the context of stochastic state reduction models (see e.g. [15] and citations therein). The results that follow on the geometric phase may hold to some extent in these more general cases also, but we shall not pursue this question here.

In our discussion we shall not confine ourselves to Berry's phase and its generalizations. In this respect there are essentially two distinct notions of the geometric phase, namely Berry's original phase, and the quantum angles [3] that in the classical limit reduce to the angles of Hannay [28]. These angles are important in relating the geometric phase acquired by a single particle to the corresponding geometric change of a coherent state that has an uncertain number of particles.

In our discussion we illustrate this distinction by means of the experiments of [40,44].

The paper is organized as follows. In Section 2 we first review the necessary background to the geometric formulation of quantum field theory, and describe the special role of the submanifold of coherent states and the geometrical features that emerge. A geometric construction that generalizes the Berry phase to the non-adiabatic case is provided in Section 3 and various examples are discussed.

This is specialized in Section 4 to cyclic evolutions that lie within the coherent state submanifold $\mathcal{C}$, for which knowledge of the intrinsic geometry of $\mathcal{C}$ is exploited to yield a general expression for the geometric phase.

In Section 5 we provide examples of the geometric phase in three situations of electromagnetism, for systems described in quantum electrodynamics (QED) by coherent states. The first example concerns the cyclic evolution of a quantum system amongst coherent states described by a closed curve in $\mathcal{C}$, and applies to the $\mathrm{SL}(2, \mathbb{C})$ spinor representation of the quantum mechanical inner product to yield a general expression for the geometric phase. Our second example discusses an experimental situation involving a discrete sequence of measurements resulting in coherent states, such that the evolution is cyclic in the single particle state space $\mathbb{P}\mathcal{H}^1$, and open in $\mathcal{C}$. As a special case we recover the classical result due to [40] for the phase shift in an electromagnetic wave passing through a sequence of polarizers, with identical initial and final polarizations, within the photon description of QED. In the third example we discuss a fibre optic experiment [44] for which the evolution is cyclic on the 2-sphere of momentum directions of the photon, in the QED description, and yet open in $\mathbb{P}\mathcal{H}^1$ and $\mathcal{C}$. The resulting rotation of the polarization vector of the photon provides an example of the quantum angles [3], which in the classical limit reduce to the classical angles of Hannay, and the relationship between this notion and Berry's phase is explained. We conclude in Section 6 with a review of the main results and the essential features of the physical examples discussed, and comment on the wider physical implications of the geometric phase for coherent states.

We shall adopt the abstract index notation throughout, as explained e.g. in [41], except where otherwise indicated by a bold rather than italicized index. In cases where the Dirac 'ket' notation is used [20], state vectors in the *single* particle Hilbert space $\mathcal{H}$ will be denoted $|\cdot\rangle$, whilst vectors in Fock space are distinguished by the *double* right angular bracket, thus $|\cdot\rangle\rangle \in \mathcal{F}$. Throughout, we maintain the distinction between state *vectors*, and their associated rays in Hilbert space which we refer to as *states*. Many of the background mathematical aspects (e.g. geometric phase, isometries, Hamiltonian flow, etc.) of the paper are covered in [35].

2. Geometric quantum theory

We shall now review some basic concepts of the geometry of quantum theory. The state manifold $\mathcal{M}$ for an $(n+1)$-dimensional quantum system consists of the rays in $\mathbb{C}^{n+1}$, which we denote $\mathbb{CP}^n$, complex projective n-space. Thus $\mathcal{M}$ has the structure of a Kähler manifold [33], and comes equipped with a symplectic tensor Ω_{ab} and Fubini-Study metric g_{ab}, both of which are Hermitian with respect to the complex structure $J_a^{\ b}$, which satisfies $J_a^{\ b} J_b^{\ c} = -\delta_a^c$, whose eigenspaces serve to define the spaces of (anti-)holomorphic tangent vectors on $\mathcal{M}$. In addition to the Hermitian property the symplectic and metrical structures are related by the complex structure according to $\Omega_{ab} = J_a^c g_{bc}$. This *compatibility* implies that the symplectic tensor Ω^{ab} with its indices raised via the (inverse) metric g^{ab} is *inverse* to Ω_{ab}, thus $\Omega^{ab}\Omega_{cb} = \delta_c^a$. The complex structure enables one to decompose a given vector as $\psi^a = \psi^\alpha \oplus \tilde{\psi}^{\alpha'}$, where ψ^α, $\tilde{\psi}^{\alpha'}$ lie in the $\pm\mathrm{i}$ (holomorphic and anti-holomorphic) eigenspaces of J respectively. For a real vector, $\tilde{\psi}^{\alpha'}$ is equal to the complex conjugate of ψ^α. In the case of a relativistic quantum field this provides the splitting of the field into its positive and negative frequency parts. In this representation the various relations above imply that the complex structure has the form $J_\alpha^\beta = \mathrm{i}\delta_\alpha^\beta$, $J_{\alpha'}^{\beta'} = -\mathrm{i}\delta_{\alpha'}^{\beta'}$, and that the symplectic structure and metric are related by $\Omega_{\alpha\beta'} = \mathrm{i}g_{\alpha\beta'}$, $\Omega_{\alpha'\beta} = -\mathrm{i}g_{\alpha'\beta}$, where $g_{\alpha\beta'}$ is itself Hermitian as a complex valued matrix (see e.g. [23] for a more detailed account of these aspects of the geometry of the state space).

In this geometrical language the *Schrödinger equation* becomes

$$\mathrm{d}\psi^a = 2\Omega^{ab}\nabla_b H\,\mathrm{d}t, \tag{2.1}$$

where the observable function H arises as the expectation of the Hamiltonian operator according to $H = \langle\psi|\hat{H}|\psi\rangle/\langle\psi|\psi\rangle$. Writing $\xi^a = \mathrm{d}\psi^a/\mathrm{d}t$, we deduce from (2.1) that $\nabla_c\xi^a = 2\Omega^{ab}H_{bc}$. Thus, by the Hermitian property of the Hamiltonian with respect to the complex structure $H_{ab} = J_a^e J_b^f H_{ef}$, and $J_b^a J_c^b = -\delta_c^a$, we see that ξ^a satisfies *Killing's equation*, $\nabla_{(a}\xi_{b)} = 0$. Accordingly ξ^a generates isometries of the state space, with respect to the Fubini-Study metric. This corresponds to the well-known unitary evolution in the Hilbert space.

In the field theoretic context we represent a general state vector as an element of *Fock space* $\mathcal{F}$:

$$|\psi\rangle\rangle = (\psi, \psi^\alpha, \psi^{\alpha\beta}, \ldots) \in \mathcal{F}, \tag{2.2}$$

where $\psi^\alpha \in \mathcal{H}^1$, $\psi^{\alpha\beta} \in \mathcal{H}^2$, etc., $\mathcal{H}^n$ is the n-particle Hilbert space [45] and the constituent tensors are symmetric, anti-symmetric for bosonic and fermionic fields respectively. The squared Hilbert space norm of a vector $|\psi\rangle\rangle \in \mathcal{F}$ is then evaluated according to

$$\langle\langle\psi|\psi\rangle\rangle = \psi\bar{\psi} + \psi^\alpha\bar{\psi}_\alpha + \psi^{\alpha\beta}\bar{\psi}_{\alpha\beta} + \cdots. \tag{2.3}$$

For a bosonic field, the creation and annihilation operators, acting on $|\psi\rangle\rangle \in \mathcal{F}$, are defined according to

$$\hat{C}_\alpha\sigma^\alpha|\psi\rangle\rangle = \hat{C}(\sigma)|\psi\rangle\rangle = (0, \sigma^\alpha\psi, \sqrt{2}\sigma^{(\alpha}\psi^{\beta)}, \sqrt{3}\sigma^{(\alpha}\psi^{\beta\gamma)}, \cdots), \tag{2.4}$$

$$\hat{A}^\alpha\bar{\sigma}_\alpha|\psi\rangle\rangle = \hat{A}(\bar{\sigma})|\psi\rangle\rangle = (\psi^\mu\bar{\sigma}_\mu, \sqrt{2}\psi^{\mu\alpha}\bar{\sigma}_\mu, \sqrt{3}\psi^{\mu\alpha\beta}\bar{\sigma}_\mu, \cdots) \tag{2.5}$$

and satisfy the commutation relations $[\hat{A}^\alpha, \hat{C}_\beta] = \delta_\beta^\alpha$. In terms of the (Hermitian) field and momentum operators $\hat{\Phi}$, $\hat{\Pi}$ the annihilation operator is given by $\hat{A}(x) = \hat{\Phi}(x) + \mathrm{i}\hat{\Pi}(x)$ and the above commutation relations are equivalent to the canonical commutation relations $[\hat{\Phi}(x), \hat{\Pi}(x')] = \mathrm{i}\delta(x - x')$ where x, x' denote points in spacetime.

We consider the *universal bundle* (see e.g. [26]) $\mathcal{U}$ over the projective Hilbert space $\mathbb{P}\mathcal{H}$, with projection $\varpi : \mathcal{U} \to \mathbb{P}\mathcal{H}$, defined so that the fibre above any element of $\mathbb{P}\mathcal{H}$ is the ray in the Hilbert space $\mathcal{H}$ that it represents. A tangent vector $\mathrm{d}|\psi(\lambda)\rangle/\mathrm{d}\lambda$ can be decomposed into horizontal and vertical parts respectively by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}|\psi(\lambda)\rangle = \frac{\delta}{\mathrm{d}\lambda}|\psi(\lambda)\rangle + \frac{\mathrm{D}}{\mathrm{d}\lambda}|\psi(\lambda)\rangle \tag{2.6}$$

with vertical part

$$\mathrm{D}|\psi(\lambda)\rangle = \frac{\langle\psi(\lambda)|\mathrm{d}|\psi(\lambda)\rangle}{\langle\psi(\lambda)|\psi(\lambda)\rangle}|\psi(\lambda)\rangle \tag{2.7}$$

(cf. [6]). The *Fubini-Study* metric on the *projective* Hilbert space $\mathbb{P}\mathcal{H}$ is then determined by the horizontal tangent vector according to

$$\mathrm{d}s^2 = 4\frac{\langle\delta\psi(\lambda)|\delta\psi(\lambda)\rangle}{\langle\psi|\psi\rangle} = 4\left\{\frac{\langle\mathrm{d}\psi|\mathrm{d}\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\psi|\mathrm{d}\psi\rangle\langle\mathrm{d}\psi|\psi\rangle}{\langle\psi|\psi\rangle^2}\right\}, \tag{2.8}$$

where the second equality follows from (2.6). In homogeneous coordinates ψ^α on $\mathbb{CP}^n$, (2.8) becomes

$$\mathrm{d}s^2 = 8\frac{\psi^\alpha\,\mathrm{d}\psi^\beta\bar{\psi}_{[\alpha}\,\mathrm{d}\bar{\psi}_{\beta]}}{(\psi^\gamma\bar{\psi}_\gamma)^2}. \tag{2.9}$$

An identical argument applies in the case of Fock space for which $|\psi\rangle\rangle \in \mathcal{F}$.

2.1. Geometry of coherent states

The *coherent state vectors* $\{|\psi_\mathrm{c}\rangle\rangle \in \mathcal{F}\}$, for a bosonic field, are generated via the exponential map $\mathcal{E} : |\psi\rangle \mapsto |\psi_\mathrm{c}\rangle\rangle$ (cf. [24]) according to

$$|\psi_\mathrm{c}\rangle\rangle = \exp(\psi^\alpha\hat{C}_\alpha)|0\rangle\rangle = (1, \psi^\alpha, \psi^\alpha\psi^\beta/\sqrt{2!}, \ldots, \psi^\alpha\psi^\beta\cdots\psi^\delta/\sqrt{n!}, \ldots) \tag{2.10}$$

and have normalization $\langle\langle\psi_\mathrm{c}|\psi_\mathrm{c}\rangle\rangle = \exp(\langle\psi|\psi\rangle)$. Observe that a coherent state vector depends on the choice of phase and amplitude of the underlying single particle state vector $\psi^\alpha \in \mathcal{H}^1$, according to (2.10). The exponentiation (2.10) can be regarded as a mapping $\mathcal{E}$ from the universal bundle into Fock space, $\mathcal{E} : \mathcal{U} \to \mathcal{F}$, which is non-constant along each fibre $\varpi^{-1}(s)$ for all single particle states $s \in \mathbb{P}\mathcal{H}^1$.

From (2.10) the actions of $\hat{A}^\alpha$, $\hat{C}_\beta$ on $|\psi_\mathrm{c}\rangle\rangle$ are

$$\hat{A}^\alpha|\psi_\mathrm{c}\rangle\rangle = \psi^\alpha|\psi_\mathrm{c}\rangle\rangle \leftrightarrow \langle\langle\psi_\mathrm{c}|\hat{C}_\alpha = \langle\langle\psi_\mathrm{c}|\bar{\psi}_\alpha, \tag{2.11}$$

$$\hat{C}_\alpha|\psi_\mathrm{c}\rangle = \frac{\mathrm{d}|\psi_\mathrm{c}\rangle\rangle}{\mathrm{d}\psi^\alpha} \leftrightarrow \langle\langle\psi_\mathrm{c}|\hat{A}^\alpha = \frac{\mathrm{d}\langle\langle\psi_\mathrm{c}|}{\mathrm{d}\bar{\psi}_\alpha} \tag{2.12}$$

which define the space of coherent states (e.g. [32]). Observe from (2.11) that coherent states are eigenstates of the annihilation operator.

The calculation of the geometric phase for a coherent state evolution, to follow in Section 4, requires the following result, due (in the case of single particle quantum mechanics, $\mathcal{H}^1$) to [6], and generalized to a quantum field (Fock space $\mathcal{F}$) by Field [21,22] and Field and Hughston [23].

Theorem 2.1. *The Fubini-Study metric induced on the coherent state submanifold $\mathcal{C}$ is intrinsically flat. Moreover the Hilbert space vectors $\psi^\alpha \in \mathcal{H}$ provide (complex valued) Euclidean coordinates for $\mathcal{C}$.*

Proof. We provide two independent derivations of this result for a bosonic quantum field. Firstly we demonstrate how an earlier treatment due to [6] for single particle ($\mathcal{H}^1$) quantum mechanics generalizes to Fock space, and secondly we outline the proof of the result within the context of abstract Fock space given previously in [21–23].

(a) For an n-dimensional harmonic oscillator the contribution to the coherent state in the product expansion is the Gaussian wave-packet with wave-function (see e.g. [6,32]):

$$\tilde{\psi}_{q^i(s),p_i(s)}(x) = [2\pi(\Delta q)^2]^{-n/4} \exp\left(-\frac{(\mathbf{x}-\mathbf{q}(s))^2}{(2\Delta q)^2} + \frac{\mathrm{i}}{\hbar}\mathbf{p}(s)\cdot\mathbf{x}\right) \tag{2.13}$$

which is normalized so that $\langle\tilde{\psi}|\tilde{\psi}\rangle \equiv 1$. Applying $\partial/\partial q^i$ to the normalization condition we find $\langle\tilde{\psi}|x^i|\tilde{\psi}\rangle = q^i$, whereupon $\partial/\partial q^j$ leads to $\langle\tilde{\psi}|x^i x^j|\tilde{\psi}\rangle = \delta_{ij}(\Delta q)^2 + q^i q^j$. From (2.13) the differential of the normalized coherent state vector is given by

$$|\mathrm{d}\tilde{\psi}\rangle = \left(-\frac{\mathbf{q}\cdot\mathrm{d}\mathbf{q}}{2(\Delta q)^2} + \mathbf{x}\cdot\left(\frac{\mathrm{d}\mathbf{q}}{2(\Delta q)^2} + \frac{\mathrm{i}}{\hbar}\,\mathrm{d}\mathbf{p}\right)\right)|\tilde{\psi}\rangle. \tag{2.14}$$

Thus $\mathrm{i}\langle\tilde{\psi}|\mathrm{d}\tilde{\psi}\rangle = -\mathbf{q}\cdot\mathrm{d}\mathbf{p}/\hbar$, together with $\langle\mathrm{d}\tilde{\psi}|\mathrm{d}\tilde{\psi}\rangle = \mathrm{d}\mathbf{q}^2/4(\Delta q)^2 + (\Delta q)^2\,\mathrm{d}\mathbf{p}^2/\hbar^2 + (\mathbf{q}\cdot\mathrm{d}\mathbf{p})^2/\hbar^2$. The squared *horizontal* differential is therefore given by $\langle\delta\tilde{\psi}|\delta\tilde{\psi}\rangle = \mathrm{d}\mathbf{q}^2/4(\Delta q)^2 + (\Delta q)^2\,\mathrm{d}\mathbf{p}^2/\hbar^2$. Using the Heisenberg relation $\Delta q\Delta p = (1/2)\hbar$, the metrical line element induced on $\mathcal{C}$ is therefore

$$\mathrm{d}s^2 = \frac{\mathrm{d}\mathbf{q}^2}{(\Delta q)^2} + \frac{\mathrm{d}\mathbf{p}^2}{(\Delta p)^2}. \tag{2.15}$$

In quantum field theory (e.g. QED) we have $\boldsymbol{\Psi} = \prod_{I=1}^{\infty}\psi_I$ where I indexes an individual oscillator mode and each ψ_I is a Gaussian wave-function of the above form ([32]; cf. also [46]). Accordingly the above expression for the line element induced on $\mathcal{C}$ becomes summed over all I and the flatness property is preserved.

(b) In Fock space $\mathcal{F}$ [21–23] we demonstrate that $\mathcal{C}$ is intrinsically flat independently of the choice of representation for $\psi^\alpha \in \mathcal{H}$.[1] This argument extends the result to the case of interacting fields. We assume only the commutation relations for the creation and annihilation operators, $[\hat{A}^\alpha, \hat{C}_\beta] = \delta^\alpha_\beta$, and the defining relations (2.11) and (2.12). For an interacting Lagrangian $\hat{A}^\alpha$ and $\hat{C}_\beta$ are modified; nevertheless the fundamental algebraic commutation relation between the properties (2.11) and (2.12) are preserved. The particle concept then emerges asymptotically for free fields.

The Fubini-Study metric on projective *Fock* space $\mathbb{P}\mathcal{F}$ is identical to (2.8) with $|\psi\rangle$ replaced by $|\psi\rangle\rangle \in \mathcal{F}$. Let $|\psi\rangle\rangle \in \mathcal{F}$ denote the Fock state vector that is coherent with respect to $\psi^\alpha \in \mathcal{H}$ according to (2.10). From (2.12) we find $\langle\langle\mathrm{d}\psi|\mathrm{d}\psi\rangle\rangle = \mathrm{d}\bar{\psi}_\beta\mathrm{d}\psi^\alpha\langle\langle\psi|\hat{A}^\beta\hat{C}_\alpha|\psi\rangle\rangle$. Applying the CCR $[\hat{A}^\alpha, \hat{C}_\beta] = \delta^\alpha_\beta$ this becomes $(\mathrm{d}\psi^\alpha\,\mathrm{d}\bar{\psi}_\alpha +$

[1] The latter references should be consulted for two other independent proofs of this result in the context of Fock space.

$\bar{\psi}_\alpha \psi^\beta \, \mathrm{d}\psi^\alpha \, \mathrm{d}\bar{\psi}_\beta) \langle\langle \psi|\psi\rangle\rangle$. Similarly $\langle\langle \psi|\mathrm{d}\psi\rangle\rangle = \langle\langle \psi|\hat{C}_\alpha|\psi\rangle\rangle \, \mathrm{d}\psi^\alpha = \langle\langle \psi|\psi\rangle\rangle \bar{\psi}_\alpha \, \mathrm{d}\psi^\alpha$ and therefore the induced line element on $\mathcal{C}$ reduces to the *complex Euclidean* form, parameterized by $\psi^\alpha \in \mathcal{H}^1$,

$$\mathrm{d}s^2 = 4 \, \mathrm{d}\psi^\alpha \, \mathrm{d}\bar{\psi}_\alpha. \qquad \square \tag{2.16}$$

3. Symplectic construction of the geometric phase

Let $\mathcal{P}$ denote the relevant quantum state space $\mathbb{P}\mathcal{H}$ or $\mathbb{P}\mathcal{F}$ as appropriate to the context. In the non-adiabatic case, for an arbitrary quantum system that undergoes the cyclic evolution $\gamma \subset \mathcal{P}$, the geometric phase may be characterized by the following gauge invariant symplectic integral in $\mathcal{P}$,

$$\vartheta_{\mathcal{G}}[\gamma] = \int_{S \subset \mathcal{P}} \Omega, \tag{3.1}$$

where S spans γ [5,6]. Closure of the symplectic 2-form Ω ensures that the integral is independent of S. This geometrical invariant can be re-expressed in more familiar Dirac notation, as a line integral, as follows.

Proposition 3.1.

$$\vartheta_{\mathcal{G}}[\gamma] = \mathrm{i} \oint_{\tilde{\gamma} \subset \mathcal{H}} \frac{\langle \tilde{\psi}|\mathrm{d}\tilde{\psi}\rangle}{\langle \tilde{\psi}|\tilde{\psi}\rangle}, \tag{3.2}$$

where $|\tilde{\psi}\rangle$ undergoes cyclic evolution in $\mathcal{H}$ such that $\varpi[\tilde{\gamma}] = \gamma \subset \mathcal{P}$. A corresponding result holds for $|\psi\rangle\rangle \in \mathcal{F}$. The result is independent of the choice of lift $\tilde{\gamma}$ provided this is a closed curve in $\mathcal{H}$.

Although this result is well known (e.g. [4]), we shall give now a new proof of the equivalence between (3.1) and (3.2).

Proof. The Fubini-Study metric on $\mathbb{CP}^n$ has Kähler potential (e.g. [16,37]):

$$K = 4 \log(\psi^\alpha \bar{\psi}_\alpha), \tag{3.3}$$

where $\{\psi^{\boldsymbol{\alpha}}\}$ serve as homogeneous coordinates on $\mathbb{CP}^n$. Writing the holomorphic exterior derivative $\partial = (\partial(\cdot)/\partial\psi^\alpha) \, \mathrm{d}\psi^\alpha \wedge$ we find

$$\bar{\partial} K = 4 \frac{\psi^\alpha \, \mathrm{d}\bar{\psi}_\alpha}{\psi^\gamma \bar{\psi}_\gamma} \tag{3.4}$$

and thus

$$\partial\bar{\partial} K = 4 \left(\frac{\delta^\alpha_\beta}{\psi^\gamma \bar{\psi}_\gamma} - \frac{\psi^\alpha \bar{\psi}_\beta}{(\psi^\gamma \bar{\psi}_\gamma)^2} \right) \mathrm{d}\psi^\beta \wedge \mathrm{d}\bar{\psi}_\alpha. \tag{3.5}$$

The symplectic 2-form is then $\Omega = \mathrm{i}\partial\bar{\partial}K$ (which is real since the anti-commutator $\{\partial, \bar{\partial}\}$ vanishes) can be written $\Omega = i\,\mathrm{d}(\bar{\partial}K)$. Stokes' theorem applied to (3.1) yields (3.2) as required. □

Observe that the right-hand side of (3.2) is equivalent to the same expression (cf. [39]) with a term $\langle \mathrm{d}\tilde{\psi}|\tilde{\psi}\rangle$ subtracted in the numerator of the integrand, and factor of two in the denominator, since the difference between the two expressions involves $\oint \mathrm{d}\log\langle\psi|\psi\rangle$ which vanishes owing to the single valued nature of the inner product $\langle\cdot|\cdot\rangle$.

Proposition 3.1 has the following well-known consequence (cf. [38]).

Corollary 3.2. *The geometric phase $\vartheta_{\mathcal{G}}$ defined by (3.1) is equal to the phase acquired by a state vector $|\psi_{\mathrm{h}}\rangle \in \mathcal{H}$ whose evolution is horizontal in $\mathcal{U}$ and whose associated ray undergoes cyclic evolution $\gamma \subset \mathbb{P}\mathcal{H}$. In other words, $\vartheta_{\mathcal{G}}$ is the holonomy around γ of the connection on the principal fibre bundle $\mathcal{U}$, defined so that the horizontal sections are generated by the horizontal vectors $|\psi_{\mathrm{h}}\rangle \in \mathcal{H}$ according to (2.6).*

Proof. Write $|\tilde{\psi}(t)\rangle = \exp(\mathrm{i}\alpha(t))|\psi_{\mathrm{h}}(t)\rangle$ for $\alpha(t) \in \mathbb{C}$, so that $|\psi_{\mathrm{h}}\rangle$ acquires the factor $\exp(-\mathrm{i}\Delta\alpha)$ around γ. From the identity $|\mathrm{d}\tilde{\psi}\rangle = \exp(\mathrm{i}\alpha)[|\mathrm{d}\psi_{\mathrm{h}}\rangle + \mathrm{i}|\psi_{\mathrm{h}}\rangle\,\mathrm{d}\alpha]$ we deduce $\langle\tilde{\psi}|\mathrm{d}\tilde{\psi}\rangle/\langle\tilde{\psi}|\tilde{\psi}\rangle = \mathrm{i}\,\mathrm{d}\alpha$. From (3.2), therefore, we find $\vartheta_{\mathcal{G}} = -\Delta\alpha$, and so $|\psi_{\mathrm{h}}\rangle$ acquires $\exp(\mathrm{i}\vartheta_{\mathcal{G}})$ around γ, as required ($\Delta\alpha$ and therefore $\vartheta_{\mathcal{G}}$ are real since $\langle\psi_{\mathrm{h}}|\psi_{\mathrm{h}}\rangle = \exp(2\Im\alpha)\langle\tilde{\psi}|\tilde{\psi}\rangle$ and $\langle\psi_{\mathrm{h}}|\psi_{\mathrm{h}}\rangle$ is constant by the horizontal property). □

Thus, for example, a state vector that satisfies the (non-linear) modified Schrödinger equation, introduced in [31]:

$$\mathrm{i}\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = (\hat{H} - \langle\hat{H}\rangle)|\psi\rangle \tag{3.6}$$

satisfies these conditions. This equation is non-linear on the state space, although it is linear along any given unitary trajectory on Hilbert space. In respect of Corollary 3.2 we remark on the Pancharatnam 'in-phase' criterion [10,40] according to which a pair of state vectors $|\alpha\rangle$, $|\beta\rangle$ are *in-phase* if their superposition has maximum intensity amongst all superpositions of the form $|\alpha\rangle + \exp(\mathrm{i}\theta)|\beta\rangle$, i.e. $\langle\alpha|\beta\rangle$ is real and positive. A horizontal evolution $|\psi_{\mathrm{h}}(t)\rangle$ therefore has the property that, infinitesimally, neighboring state vectors $|\psi_{\mathrm{h}}(t)\rangle$ and $|\psi_{\mathrm{h}}(t + \mathrm{d}t)\rangle$ are in-phase. On integrating around γ, however, $|\psi_{\mathrm{h}}\rangle$ acquires a global phase shift. In this way it is the *non-transitivity* of the Pancharatnam in-phase criterion that gives rise to the geometric phase.

In connection with the expression (3.1) for the Berry phase we observe the following general result which relates the symplectic construction to the metrical geometry.

Theorem 3.3. *The geometric phase is bounded above according to*

$$\vartheta_{\mathcal{G}}[\gamma] \leq \inf_{S}\,[A(S)], \tag{3.7}$$

where A denotes the area functional with respect to the Fubini-Study metric, and S spans γ. Equality is attained if and only if there is an S that is a holomorphic curve spanning γ.

Proof. Given a Kähler manifold $\mathcal{K}$, and a closed curve $\gamma \subset \mathcal{K}$ the integral of the Kähler symplectic form over a real 2-surface $S \subset \mathcal{K}$ spanning γ is less than or equal to the area of this surface as measured by the induced metric. The equality holds if and only if S is a holomorphic curve with respect to the complex structure J_a^b of $\mathcal{K}$, i.e. a one-dimensional complex manifold whose tangent space is spanned by holomorphic tangent vectors T^a satisfying the eigen-property $J_b^a T^b = iT^a$ [34].[2] The result then follows by (3.1) and letting $\mathcal{K} \cong \mathbb{CP}^n$ equipped with the Fubini-Study geometry. □

In the context of spin-$\frac{1}{2}$ systems the theorem has the following elementary consequence.

Corollary 3.4. *For a spin-$\frac{1}{2}$ system, for which the relevant state space is $\mathbb{CP}^1$, the curve γ can be spanned by a holomorphic curve given by one of two surfaces on the Riemann sphere bounded by γ.*[3] *Accordingly the metrical Fubini-Study area is equal to the integral of the symplectic 2-form. For an $(n+1)$-dimensional quantum system, the eigenstates of a time independent Hamiltonian are fixed points of the unitary motion defined by (2.1) and, by linearity, the projective line L joining a pair of distinct eigenstates is also invariant. Thus all $\gamma \subset L$ generated by Schrödinger evolution are spanned by holomorphic curves, and therefore equality is obtained in Theorem 3.3.*

A more sophisticated example is provided by a system containing two interacting spin-$\frac{1}{2}$ particles [14]. In this case the relevant state space is $\mathbb{CP}^3$, and there exists a preferred (total) spin-0 state Z with orthogonal hyperplane $\bar{Z}$ consisting of all states of total spin 1. The disentangled states form a 2-quadric Q which intersects the spin-1 plane in a conic C, consisting of all spin-1 states P with definite spin direction. For a given $P \in C$ the state P' with opposite spin is obtained as the intersection of C and its tangent $\bar{P}$ that is defined as the $\mathbb{CP}^1$ orthogonal to the state P. The spin-0 state O is then obtained as the intersection of the pair of tangents to C at P, P' thus generating the spin-1 triplet P, O, P'. This situation is shown in Fig. 1. In relation to the theorem above, for each $\gamma \subset C$, the geometric phase $\vartheta_{\mathcal{G}}$ is given by the minimal spanning area (cf. [29] for a discussion of the geometry of Fig. 1 in the context of quantum mechanical measurement).

In the general case of Theorem 3.3, consider two independent Hamiltonians $H_{(i)}$, whose Hamiltonian flows generate S according to the Schrödinger equations:

$$\mathrm{d}\psi^a_{(i)} = \Omega^{ab}\nabla_b H_{(i)}\,\mathrm{d}t_{(i)}. \tag{3.8}$$

Recall the relation [9]:

$$s = \int_P^Q (g_{ab}\,\mathrm{d}\psi^a\,\mathrm{d}\psi^b)^{1/2} = \frac{1}{\hbar}\int_P^Q \Delta H\,\mathrm{d}t \tag{3.9}$$

which relation is independent of the Hamiltonian, and is a gauge invariant expression with the physical interpretation that the state space distance provides a measure of the uncertainty in the Hamiltonian generating the evolution. We have seen that the gauge invariant

[2] The authors are grateful to Ralph Howard for this reference.

[3] The choice of surface is not physically significant since the two possibilities amount to solid angles on the sphere of ϕ and $4\pi - \phi$ (counted with opposite orientation), leaving $\exp(\mathrm{i}\vartheta_{\mathcal{G}})$ invariant.

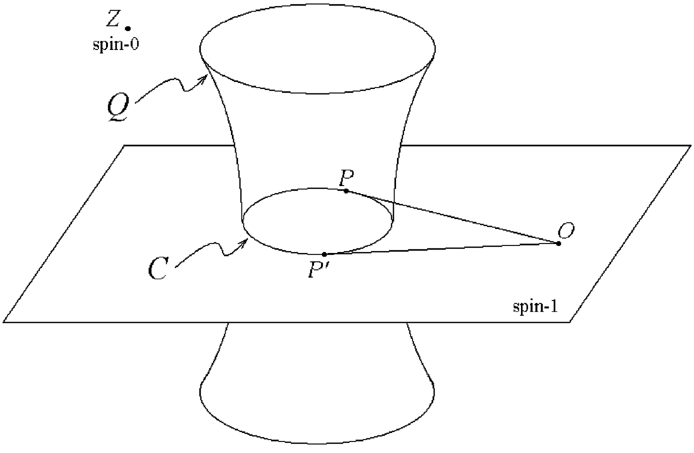

Fig. 1. System of two interacting spin-$\frac{1}{2}$ particles.

symplectic integral (3.1) manifests itself as Berry's phase for a cyclic evolution in $\mathcal{P}$, and in Section 6 we shall illustrate how this is physically observable via the principle of quantum superposition.

Likewise, the *metrical area* is manifestly gauge invariant and so admits a physical interpretation, as follows. An infinitesimal element of area is given by $\delta\mathcal{A} = |\mathrm{d}\psi_{(1)}||\mathrm{d}\psi_{(2)}|\sin\theta$ ($\psi_{(i)}$ normalized) where θ is the angle between the two state differentials with respect to the intrinsic geometry of S induced from the Fubini-Study metric. From (3.9) we have $|\mathrm{d}\psi_{(i)}| = \Delta H_{(i)}\,\mathrm{d}t_{(i)}$ together with the inner product relation $\mathrm{d}\psi^a_{(1)}\,\mathrm{d}\psi_{a(2)} = \nabla^a H_{(1)}\nabla_a H_{(2)}\,\mathrm{d}t_{(1)}\,\mathrm{d}t_{(2)}$, which follows from (3.8). Thus we find

$$\cos\theta = \frac{(\nabla^a H_{(1)})(\nabla_a H_{(2)})}{\Delta H_{(1)}\Delta H_{(2)}}. \tag{3.10}$$

The metrical area $\mathcal{A}$ generated by the pair of Hamiltonian flows is therefore given by

$$\mathcal{A} = \iint_S \sqrt{(\Delta H_{(1)})^2(\Delta H_{(2)})^2 - (\nabla_a H_{(1)}\nabla^a H_{(2)})^2}\,\mathrm{d}t_1\,\mathrm{d}t_2. \tag{3.11}$$

(This area could, if desired, also be expressed in terms of the invariant volume measure on S according to $\iint_S[\det g^{(S)}_{ab}]^{1/2}\,\mathrm{d}t_1\,\mathrm{d}t_2$.) The commutator $[\hat{H}_1, \hat{H}_2]$, in general, has an independent role from the metrical area, as follows. From (3.8), and the inverse property $\Omega_{ab}\Omega^{ac} = \delta^c_b$, we have $\Omega_{ab}\,\mathrm{d}\psi^a_{(1)}\,\mathrm{d}\psi^b_{(2)} = \Omega^{ab}\nabla_a H_{(1)}\nabla_b H_{(2)}$, which is equal to the commutator expectation function $\langle[\hat{H}_{(1)}, \hat{H}_{(2)}]\rangle$ (see e.g. [29]). Thus

$$\iint_S \Omega_{ab}\,\mathrm{d}\psi^a_{(1)}\,\mathrm{d}\psi^b_{(2)} = \iint_S \langle[\hat{H}_{(1)}, \hat{H}_{(2)}]\rangle\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{3.12}$$

and thus *the symplectic integral coincides with the integral of the commutator function of the Hamiltonian operators generating S*. Similarly for the metric, the compatibility property implies $g_{ab}\Omega^{ac}\Omega^{bd} = g^{ab}J^c_a J^d_b$, and by the Hermitian property of g_{ab} with respect to J, this is equal to g^{cd}. Thus from (3.8) the integral of the *Jordan product* (i.e. the anti-commutator) is given by the expression corresponding to (3.12) with Ω replaced by g. Thus

$$\iint_S g_{ab}\, \mathrm{d}\psi^a_{(1)}\, \mathrm{d}\psi^b_{(2)} = \iint_S \langle\{\hat{H}_{(1)}, \hat{H}_{(2)}\}\rangle\, \mathrm{d}t_1\, \mathrm{d}t_2. \tag{3.13}$$

As an illustration of this geometry where the state space is $\mathbb{CP}^1$, consider a spin-$\frac{1}{2}$ system for which S is generated by $\hat{H}_{(1)} = \sigma_z$, $\hat{H}_{(2)} = \sigma_x$ where σ_i are the SU(2) Pauli matrices. The commutator is then $[\hat{H}_{(1)}, \hat{H}_{(2)}] = 2\mathrm{i}\sigma_y$, and a spin wave-function can be written, with respect to the z-axis, as

$$|\psi\rangle = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \exp \mathrm{i}\varphi \end{pmatrix},$$

where θ, φ are standard z-polar coordinates. According to Corollary 3.4 the quantities (3.11) and (3.12) coincide for this case, and (3.12) can be verified as follows. A point on the sphere can be represented by the Cartesian vectors:

$$\mathbf{r} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) = (\cos\theta_{(x)},\ \sin\theta_{(x)}\cos\varphi_{(x)},\ \sin\theta_{(x)}\sin\varphi_{(x)}) \tag{3.14}$$

in z, x-polars respectively. From the relation $|\mathrm{d}\psi_{(i)}| = \Delta H_{(i)}\, \mathrm{d}t_{(i)}$ and the uncertainties $\Delta H_{(i)} = \sin\theta_{(i)}$ we deduce that $t_{(i)}$ are *angular* parameters, so that $\mathrm{d}t_{(i)} = \mathrm{d}\varphi_{(i)}$. The commutator function of the generating Hamiltonians is $\langle[\hat{H}_{(1)}, \hat{H}_{(2)}]\rangle = 2\mathrm{i}\sin\theta\sin\varphi$ and so integrand of the right-hand side of (3.12) is equal to $2\mathrm{i}\sin\theta\sin\varphi\, \mathrm{d}\varphi_{(x)}\, \mathrm{d}\varphi$. As $\psi_{(x)}$ changes for fixed $\theta_{(x)}$ we have the geometrical identity $\mathrm{d}\varphi_{(x)}\sin\varphi = \mathrm{d}\theta$.[4] The left-hand side of (3.12) is the volume form $\sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ on S, which establishes (3.12) in the case of this example.

4. Geometric phases for coherent states

Our purpose in this section is to illustrate how to apply the symplectic construction of the geometric phase above to the submanifold of coherent states $\mathcal{C}$. In doing so, we shall apply the results concerning the intrinsic geometry of $\mathcal{C}$, as supplied by Theorem 2.1.

The results of this section will assume that the state remains as a coherent state during the quantum evolution. This assumption is valid in the following three physical situations. If the field is ‘classical’ then it remains ‘classical’, i.e. a coherent state. This is the case with the experiments of [40] on classical electromagnetic waves. ‘Classical’ here means that the expectation value of the number operator is very large. Second, even a single particle may remain as a coherent state (Gaussian state) if a dense sequence of suitable measurements are made, as shown by Aharonov and Vardi [2]. Finally, a harmonic oscillator that is initially in a Gaussian or coherent state continues to remain in such a state during Schrödinger evolution.

[4] This can be shown e.g. by projecting a line element $\mathrm{d}\psi_{(x)}$ onto the x, y plane and calculating the length of its radial component. In terms of the x-azimuth this yields $\mathrm{d}\varphi_{(x)}\sin\theta_{(x)}\sin\varphi_{(x)}\sin\varphi$, which can also be measured as $\mathrm{d}\theta\cos\theta$, from which the identity follows using (3.14).

The integral of the symplectic 2-form $\Omega = \Omega_{\alpha\beta'}\,\mathrm{d}\psi^{\alpha} \wedge \mathrm{d}\bar{\psi}^{\beta'}$ over an open 2-surface S spanning the given closed curve γ is independent of S, by virtue of Stoke's theorem and the closure of Ω. Thus for a cyclic evolution amongst coherent states described by a closed curve $\Gamma \subset \mathcal{C}$ we may restrict the spanning surface S to lie within $\mathcal{C}$ in order to calculate the phase (3.2). Thus for a closed curve in $\mathcal{C}$ we have $\vartheta_{\mathcal{G}} = \iint_S \Omega_{(\mathcal{C})}$, where $\Omega_{(\mathcal{C})}$ can be taken as the induced symplectic form on $\mathcal{C}$. Theorem 2.1 implies that $\Omega_{(\mathcal{C})}$ is identical to the symplectic form $\Omega^{(\mathcal{H})}$ on the single particle Hilbert space $\mathcal{H}$, which is complex Euclidean. (The symplectic tensor $\Omega_{\alpha\beta'} = \mathrm{i}g_{\alpha\beta'}$ on the ambient space $\mathbb{P}\mathcal{F}$, evaluated at a point $P \in \mathcal{C}$, is distinct from that determined by (2.16), but reduces to this Euclidean form when evaluated on *tangent* vectors to $\mathcal{C}$, such as in the present case where $S \subset \mathcal{C}$.) Since $\Omega^{(\mathcal{H})}$ is a closed 2-form on the Kähler manifold $\mathcal{H}$, the open 2-surface integral can be transformed, via Stokes' theorem (see e.g. [33]), to the closed line integral:

$$\vartheta_{\mathcal{G}} = \oint_{\tilde{\gamma}} \Omega^{(\mathcal{H})}_{ab}\,\psi^a\,\mathrm{d}\psi^b, \tag{4.1}$$

where Γ is parameterized by the closed curve $\tilde{\gamma} \subset \mathcal{H}^1$. (This follows from the exactness relation $\Omega = \mathrm{d}(\Omega^{(\mathcal{H})}_{ab}\,\psi^a\,\mathrm{d}\psi^b)$.) In Dirac notation, the integrand above can be expressed as $2\Im\langle\psi|\mathrm{d}\psi\rangle$ and so we have the following result (cf. [2]).

Proposition 4.1. *For a cyclic evolution amongst coherent states $\Gamma \subset \mathcal{C}$ parameterized by a closed curve $\tilde{\gamma} = \{\psi^{\alpha}(s)\} \subset \mathcal{H}^1$ according to (2.10), the geometric phase acquired by the coherent state vector $|\psi_{\mathrm{c}}\rangle\rangle$ is given by*

$$\vartheta_{\mathcal{G}} = 2\Im \oint_{\gamma} \langle\psi|\mathrm{d}\psi\rangle, \tag{4.2}$$

where $|\psi\rangle$ has free normalization over $\tilde{\gamma}$ and $\Im$ denotes the imaginary part. For a field of constant intensity, $\langle\psi|\psi\rangle = k$, and therefore the corresponding expression holds with $\Im$ on the right-hand side of (2.10) replaced by a factor $-\mathrm{i}$.

Combining Propositions 3.1 and 4.1, therefore, we obtain the following correspondence between the geometric phases for cyclic evolutions in the single particle state space and the coherent state submanifold.

Proposition 4.2. *Given a cyclic evolution amongst single particle states $\gamma \subset \mathbb{P}\mathcal{H}^1$ and a corresponding coherent state evolution $\Gamma \subset \mathcal{C}$ parameterized by the closed curve $\tilde{\gamma} = \{\psi^{\alpha}(s) \in \mathcal{H}^1\}$ according to (2.10), such that $\varpi[\tilde{\gamma}] = \gamma$, and of constant intensity $\langle\psi|\psi\rangle = \langle\psi|\hat{N}|\psi\rangle$ over $\tilde{\gamma}$, the geometric phases on $\mathbb{P}\mathcal{H}^1$ and $\mathcal{C}$ are related by*

$$\vartheta_{\mathcal{G}}(\Gamma) = \langle\psi|\psi\rangle\vartheta_{\mathcal{G}}(\gamma). \tag{4.3}$$

Thus, for a coherent state evolution such that the expectation of the total number operator is unity, these two phases coincide.

Proposition 4.1 shows that the phase $\vartheta_{\mathcal{G}}$, in addition to its significance modulo 2π, contains information as to the field intensity $\Lambda \in \mathbb{R}$, through its absolute value. For consider a family of closed curves $\{\Gamma_\Lambda\} \subset \mathcal{C}$ parameterized by the expectation of the total number operator $\langle \hat{N} \rangle = \Lambda$, i.e. $\Gamma_\Lambda(\theta) = \exp(\Lambda^{1/2}\psi^\alpha(\theta)\hat{C}_\alpha)|0\rangle\rangle$ for a prescribed closed curve $\{\psi^\alpha(\theta)\} \subset \mathcal{H}$ satisfying $\langle\psi|\psi\rangle = 1$. Proposition 4.1 implies that $\vartheta_{\mathcal{G}}(\Lambda) = \Lambda\vartheta_{\mathcal{G}}(1)$, i.e. for a given closed curve in $\mathbb{P}\mathcal{H}$ the geometric phase acquired by the associated evolution in $\mathcal{C}$ is *proportional to the field intensity*. As $\Lambda \to 0$ the curve $\Gamma \subset \mathcal{C}$ contracts to the vacuum $|\mathbf{0}_{\mathrm{c}}\rangle\rangle$ and correspondingly $\vartheta_{\mathcal{G}}(\Lambda) \to 0$. In general, the coherent state vector acquires a phase factor $\exp(\mathrm{i}\Lambda\vartheta_{\mathcal{G}}(1))$ as a result of cyclic evolution around $\Gamma \subset \mathcal{C}$. This establishes the following result.

Corollary 4.3. *Let $\mathcal{W}$ denote the winding number of the locus $z_\Lambda = \{\exp(\mathrm{i}\lambda\vartheta_{\mathcal{G}}(1)) \in \mathbb{C}|0 \le \lambda \le \Lambda\}$ about the origin in the complex plane $\mathbb{C}$. Then the field intensity Λ is related to $\mathcal{W}$ according to $\Lambda \simeq 2\pi|\mathcal{W}|/\vartheta_{\mathcal{G}}(1)$. Equality is attained if and only if $\Lambda\vartheta_{\mathcal{G}}(1) = 2\pi n$ for integer n.*

5. Electromagnetic manifestation of geometric phases

We consider three examples of geometric phase involving coherent states of the electromagnetic field. First, using the state space geometry we have described, we present a general (spinor) formula for the Berry phase acquired for a cyclic evolution that is described by a closed curve in $\mathcal{C}$.

In the second example we explain a classical experiment due to [40], involving a plane polarized ‘classical’ electromagnetic wave passing through a sequence of polarizers, whose QED description is such that the evolution is cyclic in the projective single particle Hilbert space $\mathcal{P}$ and the system remains in coherent states. This is generalized to an arbitrary cyclic evolution in $\mathcal{P}$, arising from a discrete sequence of measurements.

Thirdly, we study an example of [44] involving the passage of a photon, in a coherent state, through a fibre optic medium. In this case the evolution is cyclic with respect to the principal null direction of the underlying null electromagnetic field, whose principal spinor specifies the momentum direction of the photon. In this case a certain geometric phase, distinct from Berry’s phase and akin to the classical angles of Hannay, emerges.

5.1. Cyclic evolution with respect to coherent state manifold

The electromagnetic field is described in the 2-component spinor formalism by $F_{ab} = \psi_{AB}\varepsilon_{A'B'} + \bar{\psi}_{A'B'}\varepsilon_{AB}$, and the associated energy momentum is given by the spinor product $T_{ab} = \psi_{AB}\bar{\psi}_{A'B'}$ [41]. (The field tensor F_{ab} is real valued since the photon is its own antiparticle.) The condition that the momentum $p_a = T_{ab}t^b$ be a null vector, or equivalently that the photon have a well defined momentum direction, can be expressed by the requirement that the field spinor be *null*, i.e. $\psi_{AB} = \nu_A\nu_B$. For suppose $\psi_{AB} = k\mu_{(A}\lambda_{B)}$, $k \ne 0$, with μ_A, λ_A scaled so that $t^a\mu_a = t^a\lambda_a = 1$ ($\mu^a = \mu^A\bar{\mu}^{A'}$ and likewise for λ). Then p_a is given by

$$p_b = |k|^2(t\cdot\mu)\lambda_b + (t\cdot\lambda)\mu_b + (t^{AA'}\lambda_A\bar{\mu}_{A'})\mu_B\bar{\lambda}_{B'} + (t^{AA'}\bar{\lambda}_{A'}\mu_A)\bar{\mu}_{B'}\lambda_B \tag{5.1}$$

and therefore

$$p^a p_a = 2|k|^2\lambda^a\mu_a(1 - |t^{AA'}\lambda_A\bar{\mu}_{A'}|^2). \tag{5.2}$$

Thus p^a is null if and only if $\lambda \propto \mu$, i.e. the field ψ_{AB} is null (e.g., choose a basis such that

$$t^a = (1,0,0,0),\, t^{AA'} = \left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and align λ^a along the z-axis so that $\lambda_A = (\lambda_0, 0)$. Thus $t^{AA'}\lambda_A\bar{\mu}_{A'} = \lambda_0\bar{\mu}_{0'}/\sqrt{2}$ which, together with the normalizations $t^a\mu_a = t^a\lambda_a = 1$, implies $\mu_1 = 0$ so that $\mu_A \propto \lambda_A$). In other words, for a photon of the quantized electromagnetic field to be in a state of definite momentum, the associated electromagnetic field, given by the expectation of the field operator, has *coincident* principle null directions [41]. In this case it is straightforward to verify that the momentum of the photon lies along the direction of the future pointing null vector $\nu^a \leftrightarrow \nu^A\bar{\nu}^{A'}$.

To calculate the geometric phase explicitly we require a gauge invariant expression for the quantum mechanical inner product $\langle\cdot|\cdot\rangle$ for a spin-1 zero rest-mass field that is expressed directly in terms of the principal field spinor ϕ_{AB}. In the electromagnetic case such an expression first appears (in vector-tensorial form) in [27], and was subsequently generalized to massless bosonic fields of spins 0,1,2 in [22], in terms of 2-component spinors.

Consider a single photon state described by the positive frequency field $F_{ab}^{(+)} \in \mathcal{H}^{(+)}$, with left/right-handed decomposition $F_{ab}^{(+)} = \phi_{AB}^{(+)}\varepsilon_{A'B'} + \tilde{\phi}_{A'B'}^{(+)}\varepsilon_{AB}$, where $\phi_{AB}^{(+)}$ and $\tilde{\phi}_{A'B'}^{(+)}$ are *independent* positive frequency fields. The expression for the quantum mechanical inner product between a pair of such fields [22,27] is

$$\langle\phi|\psi\rangle = \frac{1}{2\pi^2}\iint_{\Sigma_{(x)}\times\Sigma_{(y)}} \frac{\psi_{AB}^{(+)}(y)\tilde{\phi}_{A'B'}^{(-)}(x) + \tilde{\psi}_{A'B'}^{(+)}(y)\phi_{AB}^{(-)}(x)}{(x^c - y^c)(x_c - y_c)}\,\mathrm{d}^3\Sigma_{(x)}^{AA'} \otimes \mathrm{d}^3\Sigma_{(y)}^{BB'}. \tag{5.3}$$

With regard to Proposition 4.1 we deduce that the geometric phase for $\Gamma \subset \mathcal{C}$ is given by

$$\vartheta_{\mathcal{G}} = \frac{1}{\pi^2}\Im\oint_{\tilde{\gamma}\subset\mathcal{H}} \times \left\{ \iint_{\Sigma_{(x)}\times\Sigma_{(y)}} \frac{\overbrace{\tilde{\psi}_{A'B'}^{(-)}(x)\delta\psi_{AB}^{(+)}(y)}^{\text{left helicity}} + \overbrace{\psi_{AB}^{(-)}(x)\delta\tilde{\psi}_{A'B'}^{(+)}(y)}^{\text{right helicity}}}{(x^c - y^c)(x_c - y_c)}\,\mathrm{d}^3\Sigma_{(x)}^{a} \otimes \mathrm{d}^3\Sigma_{(y)}^{b} \right\}. \tag{5.4}$$

In the case of a left-handed electromagnetic field, in a coherent state with a definite momentum direction, we obtain[5]

$$\vartheta_{\mathcal{G}} = \frac{2}{\pi^2}\Im\oint_{\tilde{\gamma}\subset\mathcal{H}}\iint_{\Sigma_{(x)}\times\Sigma_{(y)}} \frac{\bar{\nu}_{A'}(x)\bar{\nu}_{B'}(x)\nu_A(y)\delta\nu_B(y)}{(x^c - y^c)(x_c - y_c)}\,\mathrm{d}^3\Sigma_{(x)}^{a} \otimes \mathrm{d}^3\Sigma_{(y)}^{b}, \tag{5.5}$$

[5] The scalar propagator $(x^a - y^a)(x_a - y_a)$ arises from the frequency splitting in the quantum mechanical inner product, as described in [22,27].

where $\nu = \nu^{(+)}$, $\psi_{AB} = \nu_A \nu_B$ and $\tilde{\gamma}$ is the horizontal lift of $\gamma \subset \mathcal{P}$, i.e. $\tilde{\gamma} = \varpi_{\mathrm{h}}^{-1}[\gamma] \subset \mathcal{U}$.

For example, consider rotation of ν^A about the z-axis, generated by a fixed $\hat{J}_z$:

$$\nu^A(x,\varphi) = \alpha(x) \begin{pmatrix} \cos\frac{1}{2}\theta \exp\left(-\frac{1}{2}\mathrm{i}\varphi\right) \\ \sin\frac{1}{2}\theta \exp\left(\frac{1}{2}\mathrm{i}\varphi\right) \end{pmatrix} \tag{5.6}$$

where the scalar function $\alpha(x)$ reflects the spacetime degrees of freedom of ψ_{AB} and is positive frequency. The spinor ν^A rotates with respect to the azimuth φ according to

$$\delta\nu^A = -\frac{1}{2}\mathrm{i}\hat{\nu}^A\delta\varphi, \quad \hat{\nu}^{\mathbf{A}} = \begin{pmatrix} \nu^{\mathbf{0}} \\ -\nu^{\mathbf{1}} \end{pmatrix}. \tag{5.7}$$

From (5.5), therefore, we find

$$\vartheta_{\mathcal{G}} = -\frac{1}{\pi^2}\Im \oint_{\tilde{\gamma}} \mathrm{i}\,\mathrm{d}\varphi \iint_{\Sigma_{(x)}\times\Sigma_{(y)}} \frac{\bar{\nu}_{A'}(x)\bar{\nu}_{B'}(x)\nu_A(y)\hat{\nu}_B(y)}{(x^c-y^c)(x_c-y_c)}\,\mathrm{d}^3\Sigma^a_{(x)} \otimes \mathrm{d}^3\Sigma^b_{(y)}. \tag{5.8}$$

The field intensity can be expressed via (5.3) and (5.6) as

$$\langle\psi|\psi\rangle = \frac{1}{4\pi^2}\iint_{\Sigma^2} \frac{\alpha^2(x)\bar{\alpha}^2(y)}{(x^c-y^c)(x_c-y_c)}\,\mathrm{d}^3x\,\mathrm{d}^3y \tag{5.9}$$

and thus from (5.6) and (5.8) we deduce

$$\vartheta_{\mathcal{G}} = -2\oint \langle\psi|\psi\rangle \cos\theta\,\mathrm{d}\varphi. \tag{5.10}$$

For a cyclic evolution in $\mathcal{C}$ we require that ψ^α of (2.10) undergoes cyclic evolution in $\mathcal{H}^1$, and thus φ passes through an angle 4π. If the field intensity is constant (5.10) yields $\vartheta_{\mathcal{G}} = -8\pi\langle\psi|\psi\rangle\cos\theta$, which scales with the field intensity, as required by Proposition 4.1. For $\theta = 0, \pi$ the evolution is a fixed point in $\mathbb{P}\mathcal{H}^1$, but nevertheless a closed curve of positive length in $\mathcal{C} \subset \mathbb{P}\mathcal{F}$, owing to the dependence of $|\psi_{\mathrm{c}}\rangle\rangle$ on the phase of ψ^α in (2.10).

The geometric phase for a closed curve $\Gamma \subset \mathcal{C}$ is observable, in principle, via the linear superposition of state vectors in Fock space $\mathcal{F}$. According to the discussion surrounding Corollary 4.2, the (incoherent) superposition $|\chi\rangle\rangle = |\psi_{\mathrm{c}}(\Lambda_1)\rangle\rangle + |\psi_{\mathrm{c}}(\Lambda_2)\rangle\rangle$ undergoes the geometric transformation $|\chi\rangle\rangle \mapsto |\chi'\rangle\rangle = \exp(\mathrm{i}\Lambda_1\vartheta_{\mathcal{G}})|\psi_{\mathrm{c}}(\Lambda_1)\rangle\rangle + \exp(\mathrm{i}\Lambda_2\vartheta_{\mathcal{G}})|\psi_{\mathrm{c}}(\Lambda_2)\rangle\rangle$ where ψ^α undergoes cyclic evolution in $\mathcal{H}^1$. The vectors $|\chi\rangle\rangle$, $|\chi'\rangle\rangle$ project to distinct states in $\mathbb{P}\mathcal{F}$, and the Dirac transition probability between the initial and final states can be calculated in terms of the intensities Λ_1, Λ_2 and geometric phase $\vartheta_{\mathcal{G}}$. In the case $\Lambda_1 = 0$ (i.e. $|\psi_{\mathrm{c}}(\Lambda_1)\rangle\rangle$ is the vacuum), $\Lambda_2 = \Lambda$, this probability reduces to

$$\mathbf{P}(\chi,\chi') = \frac{4 + 4(1+\exp\Lambda)\cos(\Lambda\vartheta_{\mathcal{G}}) + (1+\exp\Lambda)^2}{(3+\exp\Lambda)(1+\exp\Lambda + 2\cos(\Lambda\vartheta_{\mathcal{G}}))} \le 1 \tag{5.11}$$

which is unity for $\Lambda = 0$.

Superpositions of coherent states, sometimes referred to as ‘cat’ states, involved in this situation are very difficult to produce experimentally. Nevertheless, since the principle of quantum superposition is the only essential feature required, the possibility of producing such states in experimental situations should not be disregarded, and thereby the discussion above may acquire experimental as well as theoretical significance.

5.2. Cyclic evolution in single particle state space

Consider a sequence of measurements starting from an arbitrary initial coherent state, such that the states resulting from each successive measurement remain coherent. Such a procedure is the discrete analogue of a dense sequence of measurements on a quantum system such that the state remains within $\mathcal{C}$, as described in [2].

Suppose a system is in a coherent state given by $\sum_{n=0}^{\infty} C_n|n,\alpha\rangle\rangle$, with $C_n = 1/\sqrt{n!}$ and $|n,\alpha\rangle\rangle = |\alpha\rangle^{\otimes n} \in \mathcal{H}^n$ as in (2.10). Then a measurement is made, effected by $\alpha \mapsto \beta$, and described quantum mechanically in $\mathcal{H}$ by the projection $|\alpha\rangle \mapsto \langle\beta|\alpha\rangle|\beta\rangle$. If the state remains within $\mathcal{C}$, this can be described by the action of the projection operator $\sum_{n'=0}^{\infty}|n',\beta\rangle\rangle\langle\langle n',\beta|$ on our initial Fock state vector. The resulting state vector is then

$$\sum_{n'=0}^{\infty}|n',\beta\rangle\rangle\langle\langle n',\beta|\left(\sum_{n=0}^{\infty}C_n|n,\alpha\rangle\rangle\right) = \sum_{n=0}^{\infty}C_n\langle\langle n,\beta|n,\alpha\rangle\rangle|n,\beta\rangle\rangle, \tag{5.12}$$

where we have used $\langle\langle n',\beta|n,\alpha\rangle\rangle = \delta_{nn'}\langle\langle n,\beta|n,\alpha\rangle\rangle$ and the inner product coefficients above satisfy $\langle\langle n,\beta|n,\alpha\rangle\rangle = |\langle\beta|\alpha\rangle|^n \exp(\mathrm{i}n\phi_{\alpha\beta})$ where $\phi_{\alpha\beta} = \mathrm{ph}\langle\beta|\alpha\rangle$.

An example of this situation is provided by a classical experiment, due to [40], involving an incident plane polarized electromagnetic wave encountering a sequence of polarizers, such that the plane of polarization is returned to its original setting. The experiment is of interest since a net phase shift can be predicted from the classical Maxwell equations, and yet the result anticipates the geometric Berry phase of the (more physically correct) quantum theory [13]. Indeed Pancharatnam's classical result can be shown within the context of quantum electrodynamics (QED), provided one works with *coherent* states [10]. This two-fold description of the phase shift can be understood from the following correspondence principle.

Lemma 5.1. *A classical configuration $\psi^\alpha \in \mathcal{H}^1$ (e.g. the Pancharatnam classical electromagnetic wave) arises in the quantum theory (e.g. QED) as the expectation of the field operator $\hat{A}^\alpha \oplus \hat{A}^{\alpha'}$ in a coherent state. Thus if $\psi^\alpha \mapsto \exp(\mathrm{i}\alpha)\psi^\alpha$ on cyclic evolution in $\mathbb{P}\mathcal{H}^1$, the associated classical field undergoes a phase shift α.*

Proof. From (2.10), (2.4) and (2.5) it follows that the expectation $\langle\langle\psi_\mathrm{c}|(\hat{A}^\alpha + \hat{A}^{\alpha'})|\psi_\mathrm{c}\rangle\rangle = \psi^\alpha \oplus \bar{\psi}^{\alpha'}$, i.e. the underlying solution to the classical field equations (cf. also [10]). □

It is instructive to see how the situation relates to the Hilbert space geometry we have described. The evolution we consider can be considered as a closed curve $\gamma : \alpha \to \beta \to \gamma \to \cdots \to \delta \to \alpha$ in the *projective* single particle Hilbert space $\mathcal{P}$, obtained by joining sequential states via *geodesics*. According to the Pancharatnam in-phase criterion, following Corollary 3.2, the geodesic construction has a special significance, as follows [8,42].

Lemma 5.2. *Given a distinct pair of states $\alpha, \beta \in \mathbb{P}\mathcal{H}$ or $\mathbb{P}\mathcal{F}$, and the shorter geodesic $\gamma \subset \mathbb{P}\mathcal{H}$ joining α, β, the horizontal lift $\varpi_\mathrm{h}^{-1}[\gamma]$ has the property that the state vectors $|\alpha\rangle = \varpi_\mathrm{h}^{-1}(\alpha)$ and $|\beta\rangle = \varpi_\mathrm{h}^{-1}(\beta)$ are in-phase according to the Pancharatnam criterion.*

Proof. Construct the (unique) complex projective line $L \subset \mathcal{P}$ joining α, β. The geodesic γ lies on L, as follows.

The geodesic equation for an affinely parameterized geodesic $\gamma(s) \subset (\mathcal{M}, g_{ab})$ is $(\mathrm{d}^2\psi^a/\mathrm{d}s^2) + \Gamma^a_{bc}(\mathrm{d}\psi^b/\mathrm{d}s)(\mathrm{d}\psi^c/\mathrm{d}s) = 0$ where Γ is the Christoffel connection of the metric g_{ab} on $\mathcal{M}$. Given $\mathcal{M} \subset (\mathcal{N}, h)$, and $g = h^{\mathrm{ind}(\mathcal{M})}$, the condition for γ to be a geodesic with respect to $(\mathcal{N}, h)$ is therefore $(\Gamma^a_{(h)bc} - \Gamma^a_{(g)bc})(\mathrm{d}\psi^b/\mathrm{d}s)(\mathrm{d}\psi^c/\mathrm{d}s) = \alpha(s)(\mathrm{d}\psi^a/\mathrm{d}s)$ for some scalar function α, and is affinely parameterized in $\mathcal{N}$ if and only if $\alpha(s) = 0$. On a Kähler manifold, with potential K, the Christoffel connection is determined by $\Gamma^\alpha_{\beta\gamma} = g^{\alpha\delta'}\partial^3 K/\partial\psi^\beta\partial\psi^\gamma\partial\bar{\psi}^{\delta'}$ and its complex conjugate, while all other components of Γ vanish. In the case of the Fubini-Study metric on $\mathbb{CP}^n$, therefore, we find

$$\Gamma^\alpha_{(g)\beta\gamma} = 8\frac{g^{\alpha\delta'}}{\kappa^3_{(n)}}(\bar{\psi}_\beta\bar{\psi}_\gamma\psi_{\delta'} - \kappa_{(n)}\bar{\psi}_{(\beta}\delta_{\gamma)\delta'}), \tag{5.13}$$

where $\kappa_{(n)} = 1 + \sum^n \psi^\alpha\bar{\psi}_\alpha$ and ψ^α are non-homogeneous coordinates. From the identity $\Gamma^{\mathbf{1}}_{\mathbf{11}} = (8/\kappa^3_{(n)})(g^{\mathbf{1}\delta'}\bar{\psi}^2_{\mathbf{1}}\psi_{\delta'} - g^{\mathbf{11}'}\kappa_{(n)}\bar{\psi}_{\mathbf{1}})$ on $\mathbb{CP}^n$ and $\psi^{\boldsymbol{\alpha}} = (\psi^{\mathbf{1}}, 0, 0, \dots)$ on $\gamma \subset L$, we deduce that $\Gamma^{\mathbf{1}}_{(g)\mathbf{11}} = \Gamma^{\mathbf{1}}_{(h)\mathbf{11}}$ along γ, so that $(\Gamma^{\mathbf{1}}_{(g)\mathbf{11}} - \Gamma^{\mathbf{1}}_{(h)\mathbf{11}})(\mathrm{d}\psi^{\mathbf{1}}/\mathrm{d}s)^2|_\gamma = 0$, i.e. the condition for $\gamma \subset L$ to be geodesic with respect to $\mathbb{CP}^n$.

Choose polar coordinates on L as in (5.6), with polar axis in the α direction. A geodesic then has φ constant, and (5.6) is a horizontal curve as θ varies. The overlap between the resulting state vectors is $\langle\alpha|\beta\rangle = \cos(1/2)\theta_\beta \geq 0$, as required by the Pancharatnam criterion. □

On the other hand, given a discrete sequence of states, it is possible to generate a cyclic evolution $\gamma \subset \mathcal{P}$ by joining sequential states by a set of *arbitrary* curves in $\mathcal{P}$. In this case, however, sequential state vectors, obtained from the horizontal lift of γ, are not necessarily in-phase. Conversely if we require sequential state vectors to be in-phase, for arbitrary γ, then $\varpi^{-1}[\gamma]$ is not horizontal in general.

Using the geodesic polygon construction, therefore, the geometric phase $\vartheta_{\mathcal{G}}$ can be calculated for $\gamma \subset \mathcal{P}$. The horizontal lift of $\varpi^{-1}[\gamma] \subset \mathcal{H}$ is an *open* curve, and our original state vector acquires an overall factor $\mathsf{F}\exp(\mathrm{i}\vartheta_{\mathcal{G}})$ where $\vartheta_{\mathcal{G}} = \phi_{\alpha\beta} + \phi_{\beta\gamma} + \cdots + \phi_{\delta\alpha}$ and $\mathsf{F} = |\langle\alpha|\beta\rangle\langle\beta|\gamma\rangle\cdots\langle\delta|\alpha\rangle|$. This fact can be exploited by exponentiating the state vector resulting from the cyclic evolution in $\mathcal{P}$ and forming its corresponding coherent state vector according to (2.10). Thus we have the following result.

Proposition 5.3. *The interference resulting from a discrete sequence of measurements on a quantum system whose state remains within $\mathcal{C}$ and whose evolution is cyclic in $\mathcal{P}$ with $|\alpha\rangle \mapsto |\alpha'\rangle = |\mathsf{F}\,\mathrm{e}^{\mathrm{i}\vartheta_{\mathcal{G}}}\alpha\rangle \in \mathcal{H}$ is given by the Dirac transition probability $|\langle\langle\alpha_\mathrm{c}|\alpha'_\mathrm{c}\rangle\rangle|^2/(\langle\langle\alpha_\mathrm{c}|\alpha_\mathrm{c}\rangle\rangle\langle\langle\alpha'_\mathrm{c}|\alpha'_\mathrm{c}\rangle\rangle)$ according to*

$$\mathcal{T} = \exp[-\langle\alpha|\alpha\rangle(1 - 2\mathsf{F}\cos\vartheta_{\mathcal{G}} + \mathsf{F}^2)]. \tag{5.14}$$

Then $\vartheta_{\mathcal{G}}$ is the (Berry) geometric phase acquired in $\mathcal{P}$ and coincides with the 'classical' phase shift observed by Pancharatnam; the quantity F^2 is the factor by which the intensity of the field is scaled.

In this case the relevant state space is the Poincaré sphere of polarization states, for fixed 4-momentum p^a, which is *isomorphic to the state space for a spin-$\frac{1}{2}$ particle*. The phase shift $\vartheta_{\mathcal{G}}$ in the 'classical' electromagnetic wave is therefore equal to *half* the solid angle subtended by the geodesic polygon on the Poincaré sphere.

5.3. Cyclic evolution in momentum

An experiment of Tomita and Chiao describes the passage of a photon through a fibre optic medium, with changing momentum direction due to the curvature of the medium [44]. Suppose that the quantum electrodynamic field is described by a coherent state vector $|\psi_{\mathrm{c}}\rangle\rangle$, as in (2.10), throughout the passage of the photon along the fibre. This assumption is consistent with the uncertainty principle applied to the photon along its trajectory. Accordingly the Heisenberg inequality is saturated, and Δx, Δp are constants of the motion. Correspondingly in terms of the geometry of the universal bundle $\mathcal{U}$ over the projective Fock space $\varpi : \mathcal{U} \to \mathbb{P}\mathcal{F}$ the trajectory is horizontal in $\mathcal{U}$, as explained e.g. in [23].[6] (The single particle state vector is a *free* electromagnetic field in that the charge-current 4-vector vanishes inside and outside the fibre optic medium; accordingly it is described by a solution to the homogeneous Maxwell equations. The field is nevertheless constrained by certain boundary conditions imposed by the geometry of the fibre.)

In the Tomita–Chiao experiment the polarization state does not return to its original value. Instead, the *momentum* of the photon undergoes cyclic evolution, and the geometrical phase that emerges provides an example of the *quantum angles* [3] that, in the classical limit, reduce to *Hannay's angles* ([28]; cf. [46]).

An arbitrary evolution Υ for the (spatial) directional part[7] of the 4-momentum of the photon can be generated in terms of its corresponding principal null direction (cf. Section 5.1). This is achieved via the action of the SU(2) spin operator in the fundamental (i.e. spin-$\frac{1}{2}$) representation, denoted $\hat{J}_{(1/2)}$. As $\Upsilon(\theta)$ is traversed, as shown in Fig. 2, the principal null spinor ν^A satisfies

$$\mathrm{i}\frac{\partial \nu^A}{\partial \theta} = \mathbf{n}(\theta) \cdot \hat{J}_{(1/2)}\nu^A, \tag{5.15}$$

where $\mathbf{n}(\theta)$ is a unit vector defining an instantaneous axis of rotation on the 2-sphere of momentum directions. This coincides with the Schrödinger evolution of the wave-function ν^A of a spin-$\frac{1}{2}$ particle in a (unit) magnetic field aligned with $\mathbf{n}$. Consistently, the Pauli–Lubanski spin vector of the photon $s^a = (1/2)e^a_{bcd}p^bM^{cd}$ ($M^{ab} = M_0^{ab} - x^ap^b + x^bp^a$) is aligned with its 4-momentum (see e.g. [46]).

Observe that, for a given evolution Υ, there exist *infinitely* many generators $\mathbf{n}(\theta)$, as shown in Fig. 2. This multiplicity can be understood as follows. For the spinor ν^A to generate Υ it is necessary and sufficient that $\partial\nu^a(\theta)/\partial\theta$ is tangent to the prescribed $\Upsilon \subset S^2$. For a given Υ, the family of null vectors $\{\nu^a(\theta)\}$ is fixed; by energy conservation $t^a\nu_a$ is constant and so

[6] Indeed if the photon were to exist in some state of definite momentum along the fibre, this would violate the uncertainty principle $\Delta x\Delta p \geq (1/2)\hbar$, since Δx remains bounded by virtue of the geometry of the fibre optic medium.

[7] We assume energy conservation, so that for $p^a = (p^0; \mathbf{p})$, p^0 and $|\mathbf{p}|^2$ are constants of the motion.

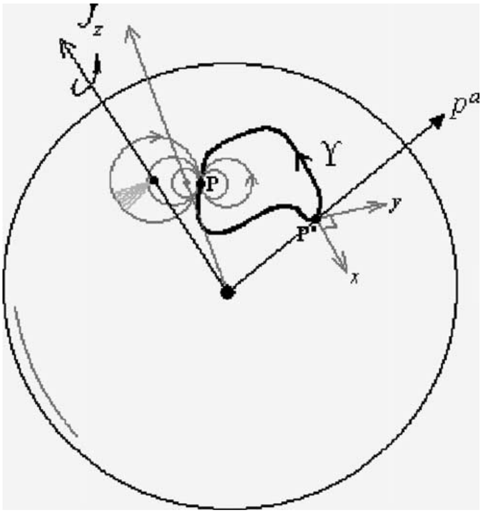

Fig. 2. Generation of rotation of spacelike 3-momentum of photon in terms of principal null direction decomposition and exponential action of SU(2) spin operator.

ν^A is subject to the freedom $\nu^A \mapsto \exp(\mathrm{i}\phi(\theta))\nu^A$ for *real* valued ϕ (under $\nu^A \mapsto \lambda\nu^A$ the energy density transforms as $T_{\mathbf{00}} \equiv \phi_{AB}\bar{\phi}_{A'B'}t^{AA'}t^{BB'} \mapsto |\lambda|^4 T_{\mathbf{00}}$ so energy conservation implies $|\lambda| = 1$).

Correspondingly the spinor derivative transforms as

$$\frac{\partial \nu^A}{\partial\theta} \mapsto \exp(\mathrm{i}\phi)\left(\frac{\partial\nu^A}{\partial\theta} + \mathrm{i}\phi'\nu^A\right) \tag{5.16}$$

which freedom preserves $\partial\nu^a/\partial\theta$. The relevant geometrical feature here is that the transformed derivative $\partial\nu^A/\partial\theta$ is not proportional to its original value, via the ϕ' term in the transformation (5.16). The freedom in the phase function $\phi(\theta)$ is reflected in that of the spin operator $\mathbf{n}_\phi(\theta)\cdot\hat{J}_{(1/2)}$, determined by the transformation (5.16) as $\mathbf{n}_\phi = \exp(\mathrm{i}\phi)[\mathbf{n} - (1/3)\phi'\hat{J}_{(1/2)}]$, which generates $\partial\nu^A/\partial\theta$ according to (5.15).

In the Tomita–Chiao experiment, for each $P \in \Upsilon$, the J axis is chosen *orthogonal* to the instantaneous p^a axis, i.e. within the (x, y)-plane, so that the corresponding circle in Fig. 2 is a *geodesic*. This property follows from the physical requirement that the triad (x, y, p) shown in Fig. 2 has zero angular velocity about the instantaneous p-axis, i.e. there is no torsion. The tangent property and orientation of Υ then fixes $\hat{J}$ uniquely. (A discussion of this geometry is given in [11], although this does not refer to the Tomita–Chiao experiment explicitly; cf. also Ref. [5] of [1].)

The spin wave-function ν^A therefore evolves according to the Schrödinger equation (5.15), such that $\langle\hat{H}\rangle = 0$ for all θ, since the ν^A spinor axis is orthogonal to the spin operator (magnetic field) direction $\mathbf{n}$. The evolution therefore coincides with that determined by the modified Schrödinger equation (3.6), so that ν^A evolves *horizontally* in $\mathcal{U}$ and acquires the geometric phase $\vartheta_{\mathcal{G}}$ around Υ. The anti-self-dual electromagnetic spinor $\psi_{AB} = \nu_A\nu_B$

therefore acquires *twice* this phase $2\vartheta_{\mathcal{G}}$, i.e. a phase factor $\exp(2i\vartheta_{\mathcal{G}})$. Its self-dual counterpart $\psi_{A'B'}$ therefore acquires the conjugate factor $\exp(-2i\vartheta_{\mathcal{G}})$, since its principal null spinor $\nu_{A'}$ lies in the conjugate spin space $\mathbb{S}_{A'}$ (cf. the factor i in (5.15)). A general polarization state can be written $|F_{ab}\rangle = z|\psi_{AB}\rangle \oplus w|\psi_{A'B'}\rangle$, and so the stereographic coordinate $q = w/z$ undergoes $q \mapsto \exp(-4i\vartheta_{\mathcal{G}})q$, which preserves the relative amplitudes of the right-handed (self-dual) and left-handed (anti-self-dual) contributions to $|F_{ab}\rangle$. Accordingly the *Stokes' vector* $p = \sqrt{q}$ undergoes $p \mapsto \exp(-2i\vartheta_{\mathcal{G}})p$, and is therefore rotated by $2\vartheta_{\mathcal{G}}$ about the axis defined by the helicity eigenstates, i.e. by the solid angle α subtended by Υ on the sphere of *momentum* directions. Since $|q|$ is preserved, the eccentricity of the ellipse of polarization is invariant, and its principal axis is rotated by α.

The case of plane polarization is $|q| = 1$, while pure helicity eigenstates correspond to $q = 0, \infty$ for which the field is said to be *circularly* polarized. In the circular case, the transformation of the Stokes vector is degenerate, and not observable at the level of the single particle Hilbert space $\mathcal{H}^1$. Nevertheless, $|F_{ab}\rangle$ acquires a pure geometric phase factor $\exp(\pm 2i\vartheta_{\mathcal{G}})$, and therefore the evolution is cyclic in $\mathbb{P}\mathcal{H}^1$. The associated classical field, according to Lemma 5.1, therefore acquires a phase shift α. Indeed, in this case, the phase shift acquired coincides with Berry's phase $\vartheta_{\mathcal{G}}^{(\psi)}$, as seen e.g. by writing the latter in terms of the tensor product $i\oint\langle\tilde{\nu}_A \otimes \tilde{\nu}_B|\tilde{\nu}_A \otimes d\tilde{\nu}_B + \tilde{\nu}_B \otimes d\tilde{\nu}_A\rangle$ which is equal to $2\vartheta_{\mathcal{G}}^{(\nu)}$. The factor of two here arises from the 2-factor principal null decomposition of the field spinor, and should be contrasted with the case of the Pancharatnam experiment described in Section 5.2 for which the solid angle α arises on the space of polarization states of the electromagnetic wave with *constant* momentum (cf. also Eq. (17) et seq. in [10]), and instead the phase shift acquired is $(1/2)\alpha$.

These results generalize to massless fields of arbitrary spin $s = (1/2)n$, as follows. A field of spin-s can be decomposed into principal spin vectors as the n-fold symmetric product $\psi_{AB\ldots E} = \lambda_{(A}\mu_B \ldots \nu_{E)}$, and in the null case $\lambda = \mu = \cdots = \nu$. In the case of constant momentum (principal null spinor) the state space is isomorphic to that of a spin-$\frac{1}{2}$ system, and thus for cyclic evolution in $\mathbb{P}\mathcal{H}^1$, the classical configuration, according to Lemma 5.1, acquires a phase shift $\frac{1}{2}\alpha$, *independent* of s.

In the case of cyclic evolution in the momentum, the anti-self-dual spinor acquires a phase factor $\exp(in\vartheta_{\mathcal{G}})$, so that the stereographic coordinate is transformed by $q \mapsto \exp(-2in\vartheta_{\mathcal{G}})q$. For arbitrary spin the Stokes' vector is given by $p = q^{1/n}$, which therefore undergoes $p \mapsto \exp(-2i\vartheta_{\mathcal{G}})p$, corresponding to a spatial rotation of $\alpha = 2\vartheta_{\mathcal{G}}$, where α is the solid angle subtended on the sphere of principal null directions $\mathbb{P}\{\nu^A\}$.[8] The result is therefore independent of s, and coincides with the angle of rotation of the triad in Fig. 2 about the p-axis, when this is parallel propagated around Υ.

On the other hand, for pure helicity states, $q = 0, \infty$, the evolution is cyclic in $\mathbb{P}\mathcal{H}^1$ and, according to Lemma 5.1, the associated classical field undergoes a phase shift $s\alpha$. For example, in the case of the (yet unobserved) graviton, for which $s = 2$, we predict a phase shift 2α. This prediction could in principle manifest itself in a classical phase shift of this

[8] In the case of the graviton (see e.g. [36]) neighboring polarization vectors lie at $\pi/4$ to each other, and the linearized metric (h_{xx}, h_{xy}) arises as two-fold tensor products of these.

amount in gravitational wave detection. Such an observation would vindicate the proposed spin-2 nature of the quantized gravitational field.

6. Conclusions

The main results of the paper are as follows. In Section 2 we have developed the quantum theory in geometric language and applied this to the space of coherent states for a bosonic quantum field. Theorem 2.1 establishes the intrinsic Euclidean geometry of the coherent state manifold in the context of both (single particle, $\mathcal{H}^1$) quantum mechanics and bosonic quantum field theory (Fock space, $\mathcal{F}$). Section 3 develops the symplectic construction of the geometric phase in a self-contained manner in terms of the Kähler geometry of the projective Hilbert space of states. Theorem 3.3 establishes the relationship between the geometric phase and the metrical area of a spanning surface for a given cyclic evolution. In Section 4 the geometric phase construction is applied to the space of coherent states. Using the result of Theorem 2.1, an expression for the geometric phase for a cyclic evolution with respect to the coherent state manifold is derived in Proposition 4.1. Comparison with the corresponding expression for single particle quantum mechanics, derived in Proposition 3.1, yields a surprising and interesting result relating the two phases by the field intensity, which is provided in Proposition 4.2.

Section 5 provides three illustrations of the geometric phase arising in situations of coherent state evolution, in the case of electromagnetism. The purpose of these examples is to demonstrate the different ways in which the *cyclic* nature of the evolution can arise. In the first example the coherent state itself, regarded as an element of the *projective* Fock space, undergoes cyclic evolution, and spinor expressions for the phase acquired by the (Fock) state vector are derived. An (incoherent) superposition of two such states, of different intensities, undergoing cyclic evolution acquires a relative geometric phase, and the transition probability amplitude between the original and resultant superpositions is calculated explicitly.

In contrast the second example concerns a situation for which the evolution is cyclic in the *single* particle state space $\mathbb{P}\mathcal{H}^1$. In this case however the coherent state after cyclic evolution is different to the original state, and the resultant corresponds to the classical phase shift in an electromagnetic wave described in [40].

The third example involves a photon (zero rest-mass particle) passing through a medium such that the *momentum* of the particle undergoes cyclic evolution. In this case, in contrast to the previous two examples, neither the single particle state nor the coherent state are restored to their original values. The result of such evolution is that the plane of polarization of the particle is rotated by the solid angle subtended on the sphere of momentum directions. These results generalize to arbitrary spin, as indicated at the end of Section 5.

In a more general context, it is worth emphasizing that our results may have significant consequences in the areas of quantum computation and cryptography, through provision of the additional ‘qubit’ of information $\vartheta_{\mathcal{G}}$. In other words, the cyclic evolution of a quantum state is able to store (some) information about the history of the state, which is encoded in the phase of the state vector. This could be exploited, for example, in designing more efficient quantum computing algorithms and in the design of quantum keys.

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